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Simple and exact series solutions for flexure of orthotropic rectangular plates with any combination of clamped and simply supported edges

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Abstract

Rectangular plates with arbitrary clamped edges are not easily amenable to exact analysis. Available solutions are either approximate or mathematically complex. The purpose of this paper to present an exact solution methodology for such problems based on superposition of double sine series solutions easily derived using the principle of virtual work. The paper includes tabulated results for two laminates for all possible combinations of clamped–simply supported edges, which would be valuable for future comparisons.

1. Introduction

Problems involving rectangular plates fall into three distinct categories:

(a) plates with all edges simply supported and subjected to arbitrary transverse load;
(b) plates with a pair of opposite edges simply supported and subjected to transverse load which is invariant along the direction of these edges;
(c) plates which do not fall into any of the above categories.

Problems of the first and second categories are amenable to straightforward rigorous analysis in terms of the well-known Navier and Levy solutions, respectively [1,2]. Such, however, is not the case with the third category for which rigorous analytical solutions, attempting to satisfy the governing differential equation and the boundary conditions exactly, turn out to be rather tedious and are hence rare; it is common practice to resort to approximate techniques based on Ritz–Galerkin methodology or direct numerical analysis. The present paper is concerned with solutions for plates with clamped–simply supported edges without sacrificing analytical rigor, and from this viewpoint, the few such analytical solutions available in the literature are reviewed here. They appear to be based on two approaches as described below.

The first approach, which has a physical appeal, is based on superposition of different solutions, each of which would involve violation of either the governing equation or some of the boundary conditions, but would be in terms of arbitrary constants which can be adjusted so that the combined solution would be exact. For example, a plate with some edges simply supported and the others clamped can be solved by superposing appropriate Levy solutions for a simply supported plate—one corresponding to the given load and the others corresponding to fixed end moments which are adjusted such that the net normal slopes are zero. Many such solutions were presented in elaborate detail by Timoshenko and Krieger [3]. Extension of the method for free-vibration studies was carried out by Gorman [4]. Superposition of other than Levy solutions is also possible; some such solutions of historical importance, for the case of a plate clamped all around, have been discussed recently [5].

The second approach, which is essentially mathematical, is based on the expansion of the deflection in a boundary discontinuous Fourier series. This was first used by Green [6] for isotropic plates, and was extended to orthotropic plates by Dickinson [7] and Whitney [8]. Since term-by-term differentiation of the series is not

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valid, the method involves a complicated procedure involving integration by parts and leads to an infinite system of equations, which can be appropriately truncated depending on the degree of accuracy desired.

The objective of the present work is to demonstrate that simple, and yet exact, series solutions can be obtained for orthotropic plates with clamped–simply supported edges. The method involves superposition of easily derived double sine series solutions and yields numerical results of any desired degree of convergence.

2. Formulation and methodology

A rectangular plate ($0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$), with some or all edges clamped and the others simply supported, is considered. The plate is assumed to be homogeneous orthotropic with the material axes coinciding with the geometric plate axes or a symmetric cross-ply laminate with perfectly bonded layers, and is subjected to arbitrary transverse load.

The present methodology is based on superposition of the following solutions for simply supported plates (Fig. 1):

(i) a Navier solution corresponding to the applied transverse load;
(ii) a number of double sine series solutions, equal to the number of clamped edges, each corresponding to the appropriate edge moment. These solutions are obtained using the principle of virtual work, which yields, as will be explicitly shown, the exact equivalent of the Levy closed-form solution for any particular harmonic of the edge moment.

Taking the moment–curvature relationship of the plate as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ & D_{22} & 0 \\ \text{sym.} & & D_{66} \end{bmatrix} \begin{Bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix} \tag{1}$$

the governing equation of the plate is given by

$$D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} = q \tag{2}$$

where $w(x,y)$ and $q(x,y)$ are the transverse deflection and the transverse load per unit area, respectively.

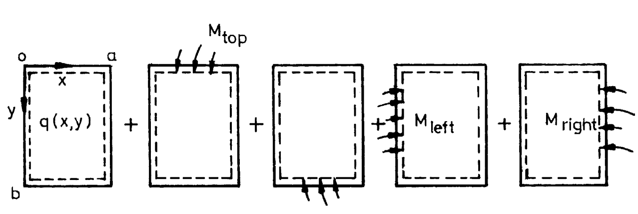

Fig. 1. Superposition of different load cases.

For simply supported boundary conditions, Navier solution results from substitution of double sine series for both q and w as given by

$$(w,q) = \sum_m \sum_n (W_{mn}, q_{mn}) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{3}$$

in the governing equation to obtain

$$W_{mn} = q_{mn} \Bigg/ \left[D_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12} + 2D_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4 \right] = q_{mn}a^2b^2/\pi^4\Gamma(m,n) \tag{4}$$

where $\Gamma(m,n) = [\lambda^2 D_{11}m^4 + 2(D_{12} + 2D_{66})m^2n^2 + D_{22}n^4/\lambda^2]$, and $\lambda = b/a$ is the aspect ratio of the plate.

Let us now consider the simply supported plate subjected to moment along an edge, say the top edge $y = 0$. Without loss of generality, this moment can be taken as

$$M_{\text{top}} = \sum_m M_{tm} \sin\frac{m\pi x}{a} \tag{5}$$

Corresponding to any particular harmonic m, the deflection is obtained using the principle of virtual work, as given by

$$\int\int [-M_x \delta w_{,xx} - M_y \delta w_{,yy} - 2M_{xy}\delta w_{,xy}]\,dx\,dy = \int_{x=0}^{a} M_{tm} \sin\frac{m\pi x}{a}(-\delta w_{,y})|_{y=0}\,dx \tag{6}$$

The solution for w is assumed, once again, in terms of double sine series as in Eq. (3). Determination of the Fourier coefficients W_{mn} is then straightforward; this merely involves the use of Eqs. (1) and (3) in Eq. (6) and the following orthogonality relations:

$$\int_{s=0}^{l} \sin\frac{\alpha\pi s}{l} \sin\frac{\beta\pi s}{l}\,ds = \int_{s=0}^{l} \cos\frac{\alpha\pi s}{l} \cos\frac{\beta\pi s}{l}\,ds = 0 \quad \text{for } \alpha \neq \beta, \quad \text{and} \quad l/2 \text{ for } \alpha = \beta \tag{7}$$

The final solution, for the particular harmonic $M_{tm} \times \sin m\pi x/a$ of the applied moment, is obtained as

$$w = \frac{2M_{tm}a^2}{\pi^3} \sin\frac{m\pi x}{a} \sum_{n=1}^{\infty} \frac{n}{\Gamma(m,n)} \sin\frac{n\pi y}{b} \tag{8}$$

It will now be shown that the above solution is an *exact* solution for the plate under the harmonic edge moment because it represents the series equivalent of the closed-form solution obtainable using Levy's method. Let us consider, for the sake of ease of presentation, a homogeneous isotropic plate with $D_{11} = D_{22} = (D_{12} + D_{66}) = D$, the flexural rigidity. Applying Levy's method, which involves reduction of the governing partial differential equation (Eq. (2)) to an ordinary differential equation by assuming a single sine series solution for w in the x-direction and determination of the Fourier coefficients by solving the ordinary differential equation along with the boundary conditions at $y = 0, b$, one readily obtains

$$w = \frac{M_{\mathrm{t}m}ab}{2Dm\pi}\sin\frac{m\pi x}{a}\left[\left(1 - \coth^2\frac{m\pi b}{a} - \frac{y}{b}\right)\sinh\frac{m\pi y}{a} + \frac{y}{b}\coth\frac{m\pi b}{a}\cosh\frac{m\pi y}{a}\right] \tag{9}$$

If the above result is expanded in a Fourier sine series in the y-direction, one gets

$$\begin{aligned} w &= \sin\frac{m\pi x}{a}\sum_{n=1}^{\infty}\frac{2M_{\mathrm{t}m}a^4b^2n}{\pi^3D(b^2m^2 + a^2n^2)^2}\sin\frac{n\pi y}{b} \\ &= \frac{2M_{\mathrm{t}m}a^2}{\pi^3D}\sin\frac{m\pi x}{a}\sum_{n=1}^{\infty}\frac{n}{(\lambda m^2 + n^2/\lambda)^2}\sin\frac{n\pi y}{b} \end{aligned} \tag{10}$$

which is the same solution as can be obtained from Eq. (8) for the isotropic homogeneous case being considered. Thus, it is clear that the principle of virtual work yields directly the series equivalent of the closed-form solution for a harmonic edge moment; this can also be verified for the orthotropic plate in a similar manner as above, though the Levy solution turns out to be rather lengthy and is hence not presented here.

The exact series solutions corresponding to sinusoidal moments applied along the bottom edge $y = b$, or the left edge $x = 0$, or the right edge $x = a$ can also be obtained using the principle of virtual work as

$$w = \frac{2M_{\mathrm{b}m}a^2}{\pi^3}\sin\frac{m\pi x}{a}\sum_{n=1}^{\infty}\frac{n\cos(n\pi)}{\Gamma(m,n)}\sin\frac{n\pi y}{b} \tag{11}$$

$$w = \frac{2M_{\mathrm{l}n}a^2}{\pi^3}\sin\frac{n\pi y}{b}\sum_{m=1}^{\infty}\frac{\lambda^2 m}{\Gamma(m,n)}\sin\frac{m\pi x}{a} \tag{12}$$

$$w = \frac{2M_{\mathrm{r}n}a^2}{\pi^3}\sin\frac{n\pi y}{b}\sum_{m=1}^{\infty}\frac{\lambda^2 m\cos(m\pi)}{\Gamma(m,n)}\sin\frac{m\pi x}{a} \tag{13}$$

where $M_{\mathrm{b}m}$, $M_{\mathrm{l}n}$ and $M_{\mathrm{r}n}$ are Fourier coefficients occurring in the following expansions:

$$\begin{aligned} M_{\mathrm{bottom}} &= \sum_m M_{\mathrm{b}m}\sin\frac{m\pi x}{a} \\ M_{\mathrm{left}} &= \sum_n M_{\mathrm{l}n}\sin\frac{n\pi y}{b} \\ M_{\mathrm{right}} &= \sum_n M_{\mathrm{r}n}\sin\frac{n\pi y}{b} \end{aligned} \tag{14}$$

The final solution for the plate with any combination of simply supported and clamped edges is obtained by superposing the exact solutions (Eqs. (3), (4), (8) and (11)–(13)), and by evaluating the Fourier coefficients $M_{\mathrm{t}m}$, etc. using the zero slope conditions at the clamped edges. These steps are illustrated here with respect to a plate clamped at the top and the left edges, and subjected to uniformly distributed load q_0. For this case, we have

$$q_{mn} = \frac{4q_0(1 - \cos m\pi)(1 - \cos n\pi)}{mn\pi^2} \tag{15}$$

The net deflection is given by

$$\begin{aligned} w = &\sum_m\sum_n \frac{4q_0a^2b^2(1 - \cos m\pi)(1 - \cos n\pi)}{mn\pi^6\Gamma(m,n)} \\ &\times \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ &+ \sum_m \frac{2M_{\mathrm{t}m}a^2}{\pi^3}\sin\frac{m\pi x}{a}\sum_{n=1}^{\infty}\frac{n}{\Gamma(m,n)}\sin\frac{n\pi y}{b} \\ &+ \sum_n \frac{2M_{\mathrm{l}n}a^2}{\pi^3}\sin\frac{n\pi y}{b}\sum_{m=1}^{\infty}\frac{\lambda^2 m}{\Gamma(m,n)}\sin\frac{m\pi x}{a} \end{aligned} \tag{16}$$

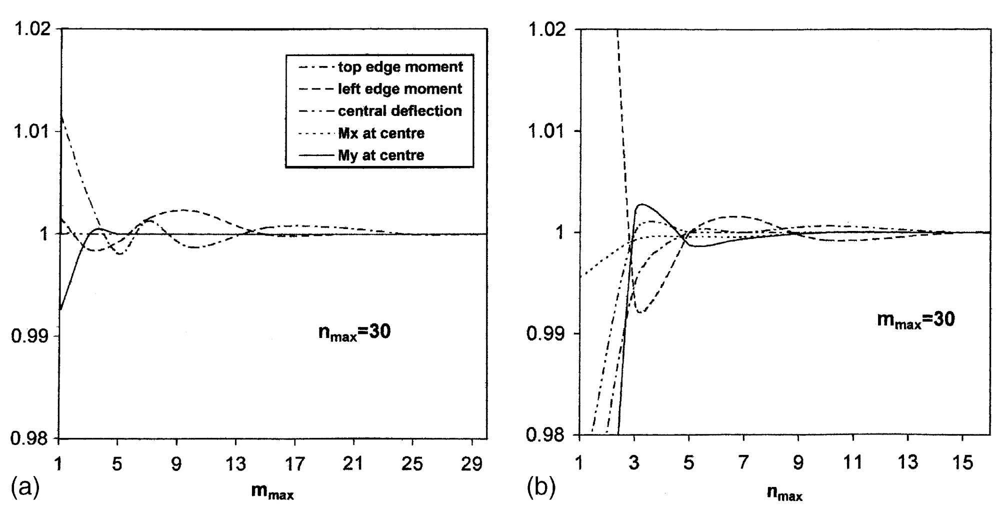

Fig. 2. Convergence of results for a (0°) plate clamped at top and left edges with respect to number of $M_{\mathrm{t}m}$'s and $M_{\mathrm{l}n}$'s (all results are normalized with respect to their final converged values).

Table 1
Deflections and moments for (0°) square plate under uniform load

Clamped edge[a]	Central deflection ($E_T h^3 w/q_0 a^4$)	Moment at center of top edge (M_y/q_0a^2)	Moment at center of left edge (M_x/q_0a^2)	Moment at center of the plate (M_x/q_0a^2)	Moment at center of the plate (M_y/q_0a^2)
None	0.006497			0.1311	0.004076
T[b]	0.006157	−0.02532		0.1244	0.005255
L	0.002644		−0.1319	0.06650	0.001028
T, B	0.005812	−0.02548		0.1176	0.006448
L, R	0.001293		−0.08602	0.04330	0.0002904
T, L	0.002615	−0.01564	−0.1310	0.06589	0.001573
T, L, R	0.001301	−0.01120	−0.08653	0.04360	0.0004897
T, L, B	0.002586	−0.01570	−0.1301	0.06528	0.002121
T, L, B, R	0.001308	−0.01121	−0.08703	0.04389	0.0006898

[a] The other edges are simply supported.
[b] T, L, B, R refer to top, left, bottom and right, respectively.

Table 2
Deflections and moments for (0°/90°/0°) square plate under uniform load

Clamped edge	Central deflection ($E_T h^3 w/q_0 a^4$)	Moment at center of top edge (M_y/q_0a^2)	Moment at center of left edge (M_x/q_0a^2)	Moment at center of the plate (M_x/q_0a^2)	Moment at center of the plate (M_y/q_0a^2)
None	0.006660			0.1298	0.008467
T	0.005999	−0.03614		0.1170	0.01025
L	0.002799		−0.1349	0.06806	0.002478
T, B	0.005340	−0.03609		0.1040	0.01204
L, R	0.001385		−0.08862	0.04481	0.0006393
T, L	0.002696	−0.02204	−0.1308	0.06563	0.003637
T, L, R	0.001378	−0.01563	−0.08839	0.04461	0.001236
T, L, B	0.002595	−0.02222	−0.1263	0.06313	0.004818
T, L, B, R	0.001371	−0.01569	−0.08809	0.04441	0.001838

Table 3
Deflections and moments for (0°) square plate under central concentrated load

Clamped edge	Central deflection ($E_T h^3 w/Pa^2$)	Moment at center of top edge (M_y/P)	Left edge (M_x/P)
None	0.02324		
T	0.02293	−0.02418	
L	0.01317		−0.5231
T, B	0.02261	−0.02433	
L, R	0.009171		−0.3959
T, L	0.01317	−0.004070	−0.5230
T, L, R	0.009169	0.002091	−0.3959
T, L, B	0.01316	−0.004086	−0.5226
T, L, B, R	0.009167	0.002090	−0.3957

Table 4
Deflections and moments for (0°/90°/0°) square plate under central concentrated load

Clamped edge	Central deflection ($E_T h^3 w/Pa^2$)	Moment at center of top edge (M_y/P)	Moment at center of left edge (M_x/P)
None	0.02131		
T	0.02040	−0.05109	
L	0.01187		−0.4494
T, B	0.01949	−0.05102	
L, R	0.008177		−0.3361
T, L	0.01179	−0.01652	−0.4464
T, L, R	0.008176	−0.002469	−0.3360
T, L, B	0.01171	−0.01668	−0.4432
T, L, B, R	0.008175	−0.002476	−0.3361

The zero slope condition at the left edge ($w_{,x} = 0$ at $x = 0$) yields

$$\left(\frac{2q_0b^2}{\pi^3}\right)\left(\frac{1-\cos n\pi}{n}\right)\sum_{m=1}^{\infty}\frac{(1-\cos m\pi)}{\Gamma(m,n)} + n\sum_{m}\frac{mM_{tm}}{\Gamma(m,n)} + \lambda^2 M_{ln}\sum_{m=1}^{\infty}\frac{m^2}{\Gamma(m,n)} = 0 \tag{17}$$

for each n.

Similarly, the zero slope condition at the top edge yields

$$\left(\frac{2q_0b^2}{\pi^3}\right)\left(\frac{1-\cos m\pi}{m}\right)\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{\Gamma(m,n)} + M_{tm}\sum_{n=1}^{\infty}\frac{n^2}{\Gamma(m,n)} + \lambda^2 m\sum_{n}\frac{nM_{ln}}{\Gamma(m,n)} = 0 \tag{18}$$

for each m.

Eqs. (17) and (18) form two sets of infinite number of equations in terms of the unknowns M_{tm} and M_{ln}, and appropriate truncation by considering a finite number of these unknowns yields a solution to the uniformly

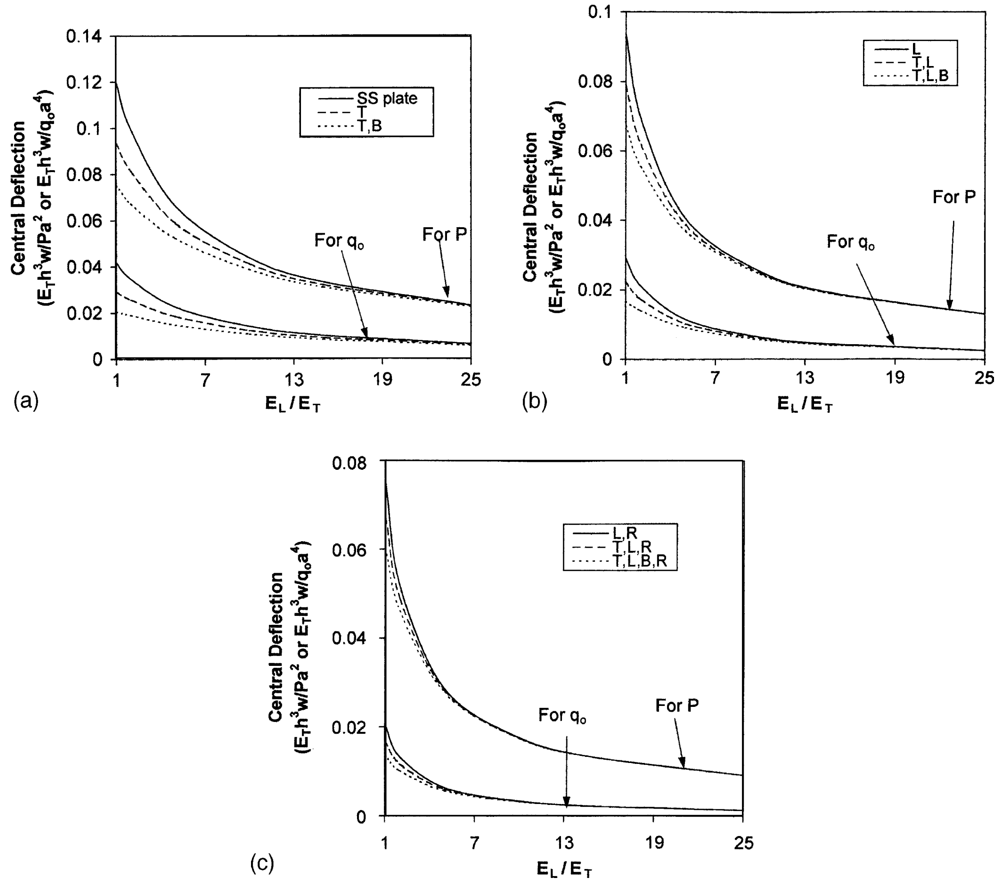

Fig. 3. Significance of top and bottom boundary conditions vs E_L/E_T for the (0°) plate.

loaded plate with two adjacent edges clamped to any desired degree of accuracy. It is easy to visualize that the more general problem would involve at the most four sets of equations, with due reduction in the number in case of any symmetry or antisymmetry about the mean lines $x = a/2$ and $y = b/2$.

3. Numerical results and discussion

Though the methodology presented above is valid for arbitrary loads, numerical results are presented here only for two important cases, viz. uniform load q_0 and a central concentrated load P. Square plates with all combinations of simply supported and clamped edges have been studied. Two lay-ups—a single layer (0°) plate and a three layer (0°/90°/0°) laminate—have been considered. The material properties have been taken to be

$$E_L/E_T = 25, \quad G_{LT}/E_T = 0.5, \quad \upsilon_{LT} = 0.25$$

where L and T are the fiber and transverse directions, respectively. The bending stiffness coefficients D_{11}, D_{12}, D_{22} and D_{66} can then be expressed in terms of E_T [9].

The number of terms to be taken in the Navier solution (Eqs. (3) and (4)) or the series for the edge moments (Eqs. (5) and (14)) has been decided based on a convergence study. While enforcing the zero slope condition at the clamped edges, the infinite series that occur in the corresponding equations (see Eqs. (17) and (18)) have been evaluated without any truncation using MATLAB [10]; this has been done by specifying the upper limit of the summation index as infinity. The evaluation is exact since corresponding closed-form equivalents are automatically substituted in MATLAB while summing infinite series; further, as can be expected, the evaluation takes less time as compared to that for a finite partial sum. This capability of the software package to evaluate infinite sums correctly has been verified by evaluating the series solution (Eq. (10)) for the moment-loaded isotropic plate and comparing the result with that obtained from the closed-form solution of Eq. (9). After the determination of the unknown moment coefficients M_{tm}, etc., the calculation of the deflections once again involves infinite series (see Eq. (16))—these have also been evaluated without any truncation using MATLAB.

The convergence of the results with respect to the increase in the number of moment coefficients is shown in Fig. 2, for the case of the (0°) plate with top and left edges clamped; all the results are presented in a normalized form with respect to their final convergent values. Because of orthotropy, the convergence of the results does not show the same trend with respect to an increase in the number of M_{1n}'s as with that in the number of M_{tm}'s. As can be expected, a larger number of terms is required for convergence of moments than for that of deflections. From Fig. 2, it is clear that the convergence is rapid and uniform though not monotonic. Similar convergence studies have been carried out for the other cases considered here; in all cases, the number of moment coefficients taken is such that the final results are all convergent up to the last decimal as presented here.

Results for the central deflection, central moments and the clamped edge moments at mid-sides are presented in Tables 1 and 2 for uniformly loaded (0°) and (0°/90°/0°) plates, respectively. Corresponding results for the central concentrated load are presented in Tables 3 and 4; the central moments for this case are infinitely large and are hence not presented. These results would be very useful for future comparisons while judging the accuracy of various approximate/numerical methods.

A careful look at the results of Tables 1–4 indicates that they are affected only marginally by a change in the boundary conditions at the top and bottom edges, with the other boundary conditions undisturbed; the only significant change occurs for M_y at the center of the uniformly loaded plate, but for this case M_y itself is quite small compared to M_x. This is due to the degree of orthotropy of the plates considered here; this is made clear in Fig. 3 where the effect of a change in E_L/E_T on the significance of the top and bottom boundary conditions is examined for the (0°) plate, with G_{LT}/E_T and υ_{LT} kept constant as earlier (i.e. 0.5 and 0.25, respectively). This figure indicate that an increase in E_L/E_T (or the corresponding increase in D_{11}/D_{22} for the lay-ups considered here) leads to a situation where the results approach the same values irrespective of the top and bottom boundary conditions, because they tend to those for the limiting case of an infinitely long plate in the y-direction.

4. Conclusion

A simple method, based on superposition of double-sine series solutions, has been presented in this paper for the analysis of arbitrarily loaded cross-ply plates with any combination of simply-supported and clamped edges. The method is valuable in view of the fact that tables of deflections and moments cannot be presented for laminated plates as for isotropic homogeneous plates even for commonly encountered loads because the results depend on the orthotropic material properties instead of a single flexural rigidity. Thus, for every new lay-up, the analysis has to be carried out afresh. For this purpose, it is necessary to have a simple and straightforward method; the present work fulfils this requirement and the solutions presented in Eqs. (4), (8) and (11)–(13) enable one to obtain accurate results for any clamped/simply supported laminated plate without tedious analytical manipulations.

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